

Radially bounded solutions of a k -Hessian equation involving a weighted nonlinear source ^{*}

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Abstract

We consider the problem

$$\begin{cases} S_k(D^2u) = \lambda|x|^\sigma(1-u)^q & \text{in } B, \\ u < 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \quad (1)$$

where B denotes the unit ball in \mathbb{R}^n , $n > 2k$ ($k \in \mathbb{N}$), $\lambda > 0$, $q > k$ and $\sigma \geq 0$. We study the existence, uniqueness and multiplicity of negative bounded radially symmetric solutions of (1). The methodology to obtain our results is based on a dynamical system approach. For this, we introduce a new transformation which reduces problem (1) to an autonomous two dimensional generalized Lotka-Volterra system.

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1 Introduction

Let $k \in \mathbb{N}$ and let Ω be a suitable bounded domain in \mathbb{R}^n . We consider the nonlinear problem

$$\begin{cases} S_k(D^2u) = f(x, u) & \text{in } \Omega, \\ u < 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where $S_k(D^2u)$ stands for the k -Hessian operator of u and f is a given nonlinear source. Problem (2) has been studied extensively by many authors in different settings. See e.g. [9, 13, 40, 41, 43, 45, 50]. The k -Hessian operator S_k is defined as follows. Let $u \in C^2(\Omega)$, $1 \leq k \leq n$, and let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the eigenvalues of the Hessian matrix (D^2u) . Then the k -Hessian operator is given by

$$S_k(D^2u) = P_k(\Lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k},$$

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where $P_k(\Lambda)$ is the k -th elementary symmetric polynomial in the eigenvalues Λ , see e.g. [50, 51]. Note that $\{S_k : k = 1, \dots, n\}$ is a family of operators which contains the Laplace operator ($k = 1$) and the Monge-Ampère operator ($k = n$). The monograph [8] is devoted to applications of Monge-Ampère equations to geometry and optimization theory. This family of operators has been studied extensively, see e.g. [28, 44] and the references therein. Recently, this class of operators has attracted renewed interest, see e.g. [6, 22, 23, 36, 18, 48, 49, 39].

We point out that the k -Hessian operators are fully nonlinear for $k \neq 1$. Further, they are not elliptic in general, unless they are restricted to the class

$$\Phi_0^k(\Omega) = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : S_i(D^2u) \geq 0 \text{ in } \Omega, i = 1, \dots, k, u = 0 \text{ on } \partial\Omega\}. \quad (3)$$

Observe that $\Phi_0^k(\Omega)$ belongs to the class of subharmonic functions. Further, the functions in $\Phi_0^k(\Omega)$ are negative in Ω by the maximum principle, see [50]. The k -Hessian operator defined on $\Phi_0^k(\Omega)$ imposes certain geometry restrictions on Ω . More precisely, domains called admissible are those whose boundary $\partial\Omega$ satisfies the inequality

$$P_{k-1}(\kappa_1, \dots, \kappa_{n-1}) \geq 0, \quad (4)$$

where $\kappa_1, \dots, \kappa_{n-1}$ denote the principal curvatures of $\partial\Omega$ relative to the interior normal. A typical example of a domain Ω for which (4) holds is a ball. For more details we refer the interested reader to [51].

Remark 1.1. Problem (2) can be easily reformulated in order to study positive solutions under the change of variable $v = -u$, which in turn yields $S_k(D^2u) = (-1)^k S_k(D^2v)$ by the k -homogeneity of the k -Hessian operator.

Now observe that, if $u \in \Phi_0^k(\Omega)$, then the right hand side of (2) must be nonnegative. Typical examples of nonlinear terms f appearing in the literature are $f(u) = |\lambda u|^p$ (see [44]), $f(u) = \lambda e^{-u}$ (see [11, 21, 25, 32] for $k = 1$ and [28, 29, 30, 31] for $1 \leq k \leq n$) and $f(u) = \lambda(1+u)^p$ (see [7, 24, 32] for $k = 1$.) The seminal contribution on the analysis of critical values for $k = 1$ with a polynomial and exponential source was made by Joseph and Lundgren in [32]. In general, for problems of Gelfand type for $k \geq 1$, the first result ($k = n$) in the radial case is due to Clément et al. [14] and for $1 \leq k \leq n$ to Jacobsen [29]. Note that in all the examples above the nonlinear terms are independent of the variable x .

Next, for our purposes we give some general notions of solutions to (2). As usual, a classical solution (or solution) of (2) is a function $u \in \Phi_0^k(\Omega)$ satisfying the equation in (2). We recall the version of the method of super and subsolutions for (2), see [50, Theorem 3.3] for more details.

Definition 1.1. A function $u \in \Phi^k(\Omega) := \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : S_i(D^2u) \geq 0 \text{ in } \Omega, i = 1, \dots, k, \}$ is called a *subsolution* (resp. *supersolution*) of (2) if

$$\begin{cases} S_k(D^2u) \geq (\text{resp. } \leq) & f(x, u) \text{ in } \Omega, \\ u \leq (\text{resp. } \geq) & 0 \text{ on } \partial\Omega. \end{cases}$$

Note that the trivial function $u \equiv 0$ is always a supersolution.

The following concept is needed to establish a general result on the existence of solutions to problem (2).

Definition 1.2. We say that a function v is a *maximal* solution of (2) if v is a solution of (2) and, for each subsolution u of (2), we have $u \leq v$.

This notion of maximal solution was recently introduced in [39] to prove the existence of solutions to problem (2).

In this article we study problem (2) on the unit ball B of \mathbb{R}^n involving a weight and a polynomial source, i.e, let us consider the problem

$$\begin{cases} S_k(D^2u) = \lambda|x|^\sigma(1-u)^q & \text{in } B, \\ u < 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \quad (5)$$

where $\lambda \in \mathbb{R}$ is a parameter and $q > 0$. In the semilinear case ($k = 1$) problem (5) was studied via bifurcation theory in [33] and recently by a variational approach in [27]. In the fully-nonlinear case, that is $k > 1$ was recently study in [39] subject to conditions $n > 2k$ and q greater or equal to Tso's critical exponent and $\sigma = 0$ (see (18) below). In [33] for $q < \frac{n+\sigma}{n-2}$, was proved that there exists $\lambda_0 > 0$, so that the problem (5) has exactly 2, 1 or 0 positive radial solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Notice that in case $\sigma > 2$ this result cover in particular supercritical nonlinearities, i.e., the ones with $q > \frac{n+2}{n-2}$.

We recall that the k -Hessian operator in radial coordinates can be written as $S_k(D^2u) = c_{n,k} r^{1-n} (r^{n-k}(u')^k)'$, where $r = |x|$, $x \in \mathbb{R}^n$ and where $c_{n,k}$ is defined by $c_{n,k} = \binom{n}{k}/n$ being $\binom{n}{k}$ the binomial coefficient.

Next, in order to state our main result, we write (5) in radial coordinates, i.e.,

$$(P_\lambda) \quad \begin{cases} c_{n,k} r^{1-n} (r^{n-k}(u')^k)' = \lambda r^\sigma (1-u)^q, & 0 < r < 1, \\ u(r) < 0, & 0 < r < 1, \\ u'(0) = 0, u(1) = 0. \end{cases}$$

Now we introduce the space of functions Φ_0^k defined on $\Omega = (0, 1)$ as in (3), for problem (P_λ) :

$$\Phi_0^k = \{u \in C^2((0, 1)) \cap C^1([0, 1]) : (r^{n-i}(u')^i)' \geq 0 \text{ in } (0, 1), i = 1, \dots, k, u'(0) = u(1) = 0\}.$$

We note that the functions in Φ_0^k are non positive on $[0, 1]$. However, if $(r^{n-i}(u')^i)' > 0$ for all $i = 1, \dots, k$, then any function in Φ_0^k is negative and strictly increasing on $(0, 1)$. This in turn implies that, if we are looking for solutions of (P_λ) in Φ_0^k , then the parameter λ must be positive.

Definition 1.3. Let $\lambda > 0$. We say that a function $u \in C([0, 1])$ is:

- (i) a *classical solution* of (P_λ) if $u \in \Phi_0^k$ and the equation in (P_λ) holds;
- (ii) an *integral solution* of (P_λ) if u is absolutely continuous on $(0, 1]$, $u(1) = 0$, $\int_0^1 r^{n-k}(u'(r))^{k+1} dr < \infty$ and the equality

$$c_{n,k} r^{n-k}(u'(r))^k = \lambda \int_0^r s^{n-1+\sigma}(1-u(s))^q ds, \quad \text{a.a. } r \in (0, 1),$$

holds whenever the integral exists.

The concept of integral solution was introduced in [14] for a more general class of radial operators, see e.g. [14] and the references therein. The standard concept of weak solution is equivalent in this case to the notion of integral solution, see [14, Proposition 2.1].

The main goal of this paper is to describe the set of negative bounded radially symmetric solutions to (5) in terms of the parameters. Our statements contain some classical and recent results (i.e. $k \geq 1$ and $\sigma = 0$), see [32, 39], where the Emden-Fowler transformation was used to have a dynamical system, which allows to obtain the desire results. The key to establish an Emden-Fowler transformation is finding an explicit singular solution U of equation in (5) on \mathbb{R}^n , that is

$$U(x) = -|x|^{-\frac{2k+\sigma}{q-k}} \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad (6)$$

corresponding to the parameter $\lambda = \tilde{\lambda}(k, \sigma)$ (see (19)). After a straightforward (and lengthy) computation, we obtain the corresponding dynamical system associated to the Emden-Fowler transformation, which looks more complicated comparing with a quadratic dynamical system. Our approach make use of a suitable change of variable, which transforms problem (P_λ) into an equivalent two-dimensional Lotka-Volterra system. In order to illustrate this approach we use the model equation in the three-dimensional space

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -r^{2m} \phi^{p+m}, \quad r > 0, \quad (7)$$

where $p \in (\frac{1}{2}, \infty)$, $m \in (-1, \infty)$ and $p + m > 0$. Equation (7) is known as a Generalized Lane-Emden-Fowler equation [46]. We note that when $m = 0$ equation (7) becomes the classical Lane-Emden equation of index p , which arises from stellar dynamic models (see [46]). Now we consider the following change of variables

$$x = -\frac{r^{2m+1} \phi^{p+m}}{\phi'}, \quad y = -\frac{r \phi'}{\phi}, \quad (8)$$

$$r = e^t. \quad (9)$$

The change of variables (8) with $m = 0$ was introduced by Milne in the early thirties, see [34, 35] and also [11].

The authors Van den Broek and Verhulst studied equation (7) using the change of variables (8) and introduced the key change of variable (9), which transforms the equation (7) into the equivalent Lotka-Volterra system

$$\begin{cases} \frac{dx}{dt} = x[2m + 3 - x - (p + m)y], \\ \frac{dy}{dt} = y[-1 + x + y]. \end{cases}$$

This approach has been used also by other authors, see e.g. [2, 52] and recently [1, 4, 47].

The paper is organized as follows: Section 2 is devoted to establish the new change of variables to reach a Lotka-Volterra system and to establish the main results, Theorem 2.1 and Theorem 2.2. The second theorem is proved in this section. In Section 3, we prove some general results on existence and nonexistence of solutions of problem (P_λ) . In Section 4, we identify the class of our Lotka-Volterra system. In Section 5 we make the local analysis of the phase portraits. Finally, in Section 6 we give the proof of Theorem 2.1.

2 New variables and main results

We consider the radial version of problem (2), as follows

$$\begin{cases} (r^{n-k}(u')^k)' = r^{n-1} f(r, u), & 0 < r < 1, \\ u(r) < 0, & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \quad (10)$$

Suppose that the function $f(r, u)$ is of the form

$$f(r, u) = c_{n,k}^{-1} h(r)(1 - u)^q,$$

where $h \in C^1(0, 1)$ and $h \neq 0$ on $(0, 1)$. Let u be a solution of (10). We define the function $w = u - 1$. Then we see that w solves the equation

$$(r^{n-k}(w')^k)' = r^{n-1} h(r)(-w)^q, \quad r > 0. \quad (11)$$

Now we are in position to state new variables to obtain a Lotka-Volterra system, which are given by

$$x(t) = r^k \frac{h(r)(-w)^q}{(w')^k}, \quad y(t) = r \frac{w'}{-w}, \quad r = e^t, \quad (12)$$

where w' stands for dw/dr . We point out that this change of variable is well-known in case $k = 1$, see e.g. [2, 52, 1, 47]. However, in the framework of the k -Hessian operator, this transformation seems to be unknown in the literature. Further, we see that such change of variable becomes optimal for problem (11), since depending on the weight h we obtain either an autonomous or non autonomous Lotka-Volterra system. More precisely, after some calculation one can see that the couple of functions $(x(t), y(t))$ solves the following (non autonomous) Lotka-Volterra system:

$$\begin{cases} \frac{dx}{dt} = x [\rho(t) - x - qy], \\ \frac{dy}{dt} = y \left[-\frac{n-2k}{k} + \frac{x}{k} + y \right], \end{cases} \quad (13)$$

where $\rho(t) = n + r \frac{h'(r)}{h(r)}$.

Now, in order to transform the problem (P_λ) into a Lotka-Volterra system (13), we set $h(r) = c_{n,k}^{-1} \lambda r^\sigma$, obtaining the autonomous dynamical system:

$$\begin{cases} \frac{dx}{dt} = x [n + \sigma - x - qy], \\ \frac{dy}{dt} = y \left[-\frac{n-2k}{k} + \frac{x}{k} + y \right]. \end{cases} \quad (14)$$

Note that we can recover w via the formula

$$w = - \left[r^{2k} h(r) \right]^{-\frac{1}{q-k}} (xy^k)^{\frac{1}{q-k}}. \quad (15)$$

We note that the existence and multiplicity results for (P_λ) are strictly related to the behavior of solutions of this dynamical system by (15). The advantage to transform problem (P_λ) into a quadratic dynamical system (14) lies in the fact that such systems has been extensively studied, see e.g. [16, 3, 12, 38]. Even when there is no a complete classification for general quadratic systems, we found a classification of phase portraits according to the space of coefficients for the particular class of Lotka-Volterra system (14), see section 4 below.

On the other hand, by the definition of the new variables (12), the region of interest is $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ (a radial solution of (P_λ) is negative and its radial derivative is positive). In this region we find four critical points: $(0, 0)$, $(0, \frac{n-2k}{k})$, $(n + \sigma, 0)$, and

$$(\hat{x}, \hat{y}) := \left(\frac{q(n-2k) - (n+\sigma)k}{q-k}, \frac{2k+\sigma}{q-k} \right). \quad (16)$$

Note that according to our general assumptions, i.e. $2k < n$ and $k < q$, the first three critical points belong to \mathbb{R}_+^2 . The fourth critical point (\hat{x}, \hat{y}) belongs to the interior of \mathbb{R}_+^2 if, and only if, $q > (n + \sigma)k/(n - 2k)$. It is not difficult to see that the orbit $(x(t), y(t))$ of (14) starts from $(n + \sigma, 0)$, see section 5.

Next, in order to estate our main result we introduce a critical exponent of the Joseph-Lundgren type, defined by

$$q_{JL}(k, \sigma) = \begin{cases} k \frac{k(k+1)n - k^2(2-\sigma) + 2k + \sigma - 2\sqrt{k(2k+\sigma)[(k+1)n - k(2-\sigma)]}}{k(k+1)n - 2k^2(k+3) - 2k\sigma - 2\sqrt{k(2k+\sigma)[(k+1)n - k(2-\sigma)]}}, & n > 2k + 8 + \frac{4\sigma}{k}, \\ \infty, & 2k < n \leq 2k + 8 + \frac{4\sigma}{k}. \end{cases} \quad (17)$$

The Joseph-Lundgren exponent, i.e.,

$$q_{JL}(1, 0) = \frac{n - 2\sqrt{n-1}}{n - 4 - 2\sqrt{n-1}}$$

was introduced in [32]. We mention that the exponent $q_{JL}(1, \sigma)$ coincides with the critical exponent founded in [17] in the study of the solutions of the problem $-\Delta u = |x|^\sigma |u|^{p-1}u$ in $\Omega \subset \mathbb{R}^n (n \geq 2)$ where $p > 1, \sigma > -2$, and Ω is a suitable domain. We prove that $q_{JL}(k, \sigma)$ plays the same role as the Joseph-Lundgren exponent, that is, as soon as the critical exponent $q_{JL}(k, \sigma)$ is crossed, a drastic change in the number of bounded solutions of (5) occurs, see Theorem 2.1 below.

Another important exponent appearing in our main result is given by

$$q^*(k, \sigma) = \frac{(n+2)k + \sigma(k+1)}{n-2k}, \quad (18)$$

which is smaller than $q_{JL}(k, \sigma)$. Further $q^*(k, \sigma)$ is bigger than $(n+\sigma)k/(n-2k)$ which ensures that (\hat{x}, \hat{y}) belongs to the interior of \mathbb{R}_+^2 for all $q \geq q^*(k, \sigma)$. The value $q^*(k, 0)$ is well-known as the critical exponent in the study of the quasilinear k -Hessian operator, see [44] for more details. The exponent $q^*(1, \sigma)$ is called the Hardy-Sobolev exponent in the study of the Hardy-Hénon equation $-\Delta u = |x|^\sigma u^p$, see [37].

Now we state our main result.

Theorem 2.1. *Let $q > k$ and $n > 2k$. Let $q^*(k, \sigma)$ and $q_{JL}(k, \sigma)$ be as in (17) and (18), respectively. Then there exists $\lambda^* > 0$ such that problem (P_λ) admits a maximal bounded solution for $\lambda \in (0, \lambda^*)$, admits at least one possible unbounded integral solution for $\lambda = \lambda^*$, and it is no classical solutions for every $\lambda > \lambda^*$.*

Moreover,

(I) *If $q^*(k, \sigma) < q < q_{JL}(k, \sigma)$ and λ is close to but not equal to*

$$\tilde{\lambda}(k, \sigma) := c_{n,k} \tau_\sigma^k (n - 2k - k\tau_\sigma), \quad (19)$$

where $\tau_\sigma := \frac{2k+\sigma}{q-k}$, then (P_λ) has a large (finite) number of solutions. In addition, if $\lambda = \tilde{\lambda}(k, \sigma)$ then there exists infinitely many solutions of (P_λ) .

(II) *If $n > 2k + 8 + \frac{4\sigma}{k}$, $q \geq q_{JL}(k, \sigma)$ and $\lambda \in (0, \lambda^*)$, then there exists only one solution of (P_λ) .*

Moreover, $\lambda^ = \lambda(k, \sigma)$.*

Similar results are well-known in the literature in case $k = 1$ and $\sigma = 0$ see e.g. [32, 19]. In case $q > k$ and $\sigma = 0$ problem (P_λ) was recently studied in [39] by a suitable Emden-Fowler transformation.

Observe that the function $u(x) = 1 + U(x)$ with $x \in B \setminus \{0\}$ and U as in (6) is an integral solution of problem (5) corresponding to the parameter $\lambda = \tilde{\lambda}(k, \sigma)$ provided that $q > q^*(k, \sigma)$ holds. Note that this integral solution can be easily obtained from the constant solution (\hat{x}, \hat{y}) of (14) via the formula (15).

Note that the effect of multiplying the nonlinearity by a weight increases the critical exponents $q^*(k, 0)$ and $q_{JL}(k, 0)$ introduced in [39]. In particular, $q^*(k, 0)$ is shifted to the value $q^*(k, \sigma)$.

Now we first note that system (14) has nontrivial closed orbits in the first quadrant if, and only if, $k < q$ and

$$-\frac{n-2k}{k}(q+1) + (n+\sigma) \left(\frac{1}{k} + 1 \right) = 0, \quad (20)$$

by [10, Theorem 2.1]. Moreover, (\hat{x}, \hat{y}) is a center. Further, if $\frac{(n+\sigma)k}{n-2k} < q$, then the center (16) belongs to the interior of the first quadrant. Observe that the unique solution q of (20) is exactly $q^*(k, \sigma)$, which is the critical exponent defined in (18).

We consider the critical exponent problem

$$\begin{cases} S_k(D^2 u) = \lambda |x|^\sigma (1-u)^{q^*(k, \sigma)} & \text{in } B, \\ u < 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \quad (21)$$

The corresponding Lotka-Volterra system is given by

$$\begin{cases} \frac{dx}{dt} = x[n + \sigma - x - q^*(k, \sigma)y], \\ \frac{dy}{dt} = y[-\frac{n-2k}{k} + \frac{x}{k} + y]. \end{cases} \quad (22)$$

It is remarkable that the line

$$\frac{n-2k}{k}x + (n + \sigma)y - \frac{n-2k}{k}(n + \sigma) = 0, \quad (23)$$

which connects the critical points $(n + \sigma, 0)$ and $(0, \frac{n-2k}{k})$, is an orbit of system (22). Moreover, from (23) we can obtain an explicit solution $(x(t), y(t))$ of system (22):

$$x(t) = (n + \sigma) \frac{c}{c + e^{\frac{2k+\sigma}{2k}t}}, \quad y(t) = \frac{n-2k}{k} \frac{e^{\frac{2k+\sigma}{2k}t}}{c + e^{\frac{2k+\sigma}{2k}t}}, \quad (24)$$

where c is a positive constant. Since that $(x(t), y(t))$ is a bounded solution of (22), we may use (15) to obtain a bounded solution w of (11) with $h(r) = c_{n,k}^{-1} \lambda r^\sigma$. Indeed, replacing (24) in (15) with $q = q^*(k, \sigma)$, we get

$$w_c = -\lambda^{-\frac{n-2k}{(2k+\sigma)(k+1)}} \frac{[c \binom{n}{k} \frac{n+\sigma}{n} (\frac{n-2k}{k})^k]^{\frac{n-2k}{(2k+\sigma)(k+1)}}}{(c + r^{\frac{2k+\sigma}{k}})^{\frac{n-2k}{2k+\sigma}}}. \quad (25)$$

Note that w_c is a scalar factor of a Bliss function, see [5]. Further, note that $w_c(\cdot)$ solves the problem

$$S_k(D^2u) = \lambda|x|^\sigma(-u)^{q^*(k,\sigma)} \quad \text{in } \mathbb{R}^n, \quad (26)$$

for all $c > 0$. Now, restricting w_c to the unit ball and setting $u = 1 + w_c$ we have that u is a solution of (21) if, and only if, there exists a value $d > 0$ such that

$$\lambda(d+1)^{k+1} - \binom{n}{k} \left(\frac{n-2k}{k}\right)^k \frac{n+\sigma}{n} d^k = 0, \quad (27)$$

with $d = c^{-1}$. We can now verify by elementary calculus that (27) has either a unique solution $d = k$ if

$$\lambda = \binom{n}{k} \frac{n+\sigma}{n} \frac{(n-2k)^k}{(k+1)^{k+1}} := \mu^*(k, \sigma) \quad (28)$$

exactly two solutions (depending on λ and σ) $d_- < d_+$ if $0 < \lambda < \mu^*(k, \sigma)$ and no solutions d if $\lambda > \mu^*(k, \sigma)$.

Hence problem (21) has a solution if, and only if, $\lambda \leq \mu^*(k, \sigma)$. If $\lambda < \mu^*(k, \sigma)$, there exist exactly two solutions of (21) given by

$$v_{\lambda,\sigma} = w_{d_-}|_B, \quad V_{\lambda,\sigma} = w_{d_+}|_B. \quad (29)$$

On the other hand, if $\lambda = \mu^*(k, \sigma)$, we have a unique solution of (21) given by

$$V_\sigma^* = w_k|_B. \quad (30)$$

Therefore, we conclude that problem (21) has exactly two solutions $u_{\lambda,\sigma}, U_{\lambda,\sigma}$ if $\lambda < \mu^*(k, \sigma)$ and a unique solution $u_\sigma^*(k, \sigma)$ if $\lambda = \mu^*(k, \sigma)$, where

$$u_{\lambda,\sigma}(x) = 1 - \lambda^{-\frac{n-2k}{(2k+\sigma)(k+1)}} (-v_{\lambda,\sigma}(x)), \quad U_{\lambda,\sigma}(x) = 1 - \lambda^{-\frac{n-2k}{(2k+\sigma)(k+1)}} (-V_{\lambda,\sigma}(x)).$$

and

$$u_\sigma^*(x) = 1 - (\mu^*(k, \sigma))^{-\frac{n-2k}{(2k+\sigma)(k+1)}} (-V_\sigma^*(x)) = 1 - \left(\frac{1+k}{1+k|x|^{\frac{2k+\sigma}{k}}} \right)^{\frac{n-2k}{2k+\sigma}}.$$

Thus we have proved the following result.

Theorem 2.2. *Let $n > 2k$. Consider the functions $v_{\lambda,\sigma}$, $V_{\lambda,\sigma}$, and V_σ^* as in (29) and (30), respectively. Let $\mu^*(k, \sigma)$ as in (28).*

(i) *If $\lambda \in (0, \mu^*(k, \sigma))$, then there exist exactly two solutions of (21) given by*

$$u_{\lambda,\sigma}(x) = 1 - \lambda^{-\frac{n-2k}{(2k+\sigma)(k+1)}}(-v_{\lambda,\sigma}(x)), \quad U_{\lambda,\sigma}(x) = 1 - \lambda^{-\frac{n-2k}{(2k+\sigma)(k+1)}}(-V_{\lambda,\sigma}(x)).$$

(ii) *If $\lambda = \mu^*(k, \sigma)$, then (21) has a unique solution given by*

$$u_\sigma^*(x) = 1 - (\mu^*(k, \sigma))^{-\frac{n-2k}{(2k+\sigma)(k+1)}}(-V_\sigma^*(x)) = 1 - \left(\frac{1+k}{1+k|x|^{\frac{2k+\sigma}{k}}} \right)^{\frac{n-2k}{2k+\sigma}}.$$

Notice that, for $q = q^*(k, \sigma)$, we have the estimate $\lambda^* \geq \mu^*(k, \sigma)$ by Theorem 2.1 (see above). We mention that, in the case $k = 1$ and $\sigma = 0$, the value $\mu^*(1, 0) = \frac{n(n-2)}{4}$ coincides with the extremal value obtained in the classical paper [32]. See also [24] and [26]. In case $\sigma = 0$, the value $\mu^*(k, 0)$ coincides with the value obtained in [39].

3 Existence and nonexistence of solutions of problem (P_λ)

We first note that equation (11) has the following scaling invariance property: let ϑ be a solution of (11) satisfying the initial conditions $\vartheta(0) = -1$, $\vartheta'(0) = 0$, then the function $\vartheta_a(r) := a^\delta \vartheta(ar)$ with $\delta = (2k + \sigma)/(q - k)$ it is also a solution of (11) for all $a > 0$.

Let u be a bounded solution of (10) with $f(r, u) = c_{n,k}^{-1} \lambda r^\sigma (1 - u)^q$ and let $u(0) = A \in (-\infty, 0)$. Let $w = u - 1$ be a solution of (11). We make a normalization by $v = w/(1 - A)$ and the rescaling

$$s = \left(\frac{\lambda}{\bar{\lambda}(k, \sigma)} \right)^{\frac{1}{2k+\sigma}} (1 - A)^{\frac{q-k}{2k+\sigma}} r, \quad r > 0.$$

We obtain that v solves the initial value problem

$$\begin{cases} (s^{n-k}(v')^k)' = s^{n-1} \bar{\lambda}(k, \sigma) s^\sigma (-v)^q, & s > 0, \\ v(0) = -1, v'(0) = 0, \end{cases} \quad (31)$$

by the boundary conditions of (10) and $\bar{\lambda}(k, \sigma) := c_{n,k}^{-1} \bar{\lambda}(k, \sigma)$. Note that by the scaling invariance property is it enough focused our study on (31). Further, the change of variable (12) applied to (31) yields with the same Lotka-Volterra system (14). Moreover we can write v in terms of the solution (x, y) of system (14) and the new parameters, that is

$$v = - [s^{2k+\sigma} \bar{\lambda}(k, \sigma)]^{-\frac{1}{q-k}} (xy^k)^{\frac{1}{q-k}}. \quad (32)$$

We will need the following two technical lemmas.

Lemma 3.1. *Let $q \geq q^*(k, \sigma)$. Then there exists a unique global solution v of (31) in the regularity class $C^2(0, \infty) \cap C^1[0, \infty)$.*

Proof. Let $\tau = \bar{\lambda}^{\frac{1}{2k+\sigma}} s$ and set $z(\tau) = v(s)$. Then we may rewrite (31) as

$$\begin{cases} (\tau^{n-k}(z')^k)' = \tau^{n-1+\sigma} (-z)^q, & \tau > 0, \\ z(0) = -1, \\ z'(0) = 0. \end{cases} \quad (33)$$

Defining $B(r) = \int_0^r s^{\frac{(\alpha-\beta)(q+1)}{\beta+1}} (a(s)s^\theta)' ds$, $r > 0$, with $\alpha = n - k$, $\beta = k$, $\gamma = n - 1$, $a(s) = s^\sigma$ and $\theta = [(\gamma+1)(\beta+1) - (\alpha-\beta)(q+1)]/(\beta+1)$, we obtain $B(r) \leq 0$ for $r > 0$ if, and only if, $q \geq q^*(k, \sigma)$. Then the global existence of (33) follows from [15, Theorem 4.1]. Now, for the uniqueness we define the map $T : \mathcal{B} \rightarrow \mathcal{B}$, where $\mathcal{B} := \{z \in C[0, t_0] : z(0) = -1 \text{ and } |z + 1| \leq 1/2\}$, by

$$T(z)(r) := -1 + \int_0^r \left(\frac{1}{t^{n-k}} \int_0^t s^{n-1+\sigma} (-z(s))^q ds \right)^{\frac{1}{k}} dt, \quad r \in [0, t_0].$$

Using the arguments given in the proof of [39, Lemma 4.1], we see that T admits a unique fixed point by the contraction mapping principle. \square

Lemma 3.2. *Let $n > 2k$, $q > k$ and $\lambda_0 > 0$. Assume that there exists a classical solution of*

$$\begin{cases} c_{n,k} r^{1-n} (r^{n-k} (w')^k)' = \lambda_0 r^\sigma (1-w)^q, & 0 < r < 1, \\ w < 0, & 0 < r < 1, \\ w'(0) = 0, w(1) = 0, \end{cases} \quad (34)$$

Then, for any $\lambda \in (0, \lambda_0)$, problem (P_λ) has a maximal bounded solution. Moreover, the maximal solutions form a decreasing sequence as λ increases.

Proof. Fix $\lambda \in (0, \lambda_0)$ and define the functions

$$g(t) = [\lambda_0(1+t)^q]^{1/k} \quad \text{and} \quad \tilde{g}(t) = [\lambda(1+t)^q]^{1/k}, \quad \text{for all } t \geq 0.$$

Set $\Phi(s) = \tilde{h}^{-1}(h(s))$ ($s \leq 0$) with h and \tilde{h} given by

$$h(s) = \int_s^0 \frac{1}{g(-t)} dt \quad \text{and} \quad \tilde{h}(s) = \int_s^0 \frac{1}{\tilde{g}(-t)} dt, \quad s \leq 0.$$

Since $q > k$, we have that $\lim_{s \rightarrow -\infty} h(s)$ exists and hence Φ is bounded by [39, Lemma 2.1 (i)-(ii)]. Next, by (34) and the convexity of Φ by [39, Lemma 2.1 (iii)] we have

$$\begin{aligned} S_k(D^2\Phi(w)) &= c_{n,k} k r^{1-k} (\Phi'(w)w')^{k-1} \left(\Phi''(w)(w')^2 + \Phi'(w)w'' + \frac{n-k}{k} \frac{\Phi'(w)w'}{r} \right) \\ &\geq c_{n,k} k r^{1-k} (\Phi'(w))^k (w')^{k-1} \left(w'' + \frac{n-k}{k} \frac{w'}{r} \right) \\ &= (\Phi'(w))^k S_k(D^2w) = \frac{(\tilde{g}(-\Phi(w)))^k}{(g(-w))^k} S_k(D^2w) = \lambda r^\sigma (1 - \Phi(w))^q. \end{aligned}$$

Therefore, $\Phi(w)$ is a bounded subsolution of (P_λ) and hence by the method of super and subsolutions we have, by [50, Theorem 3.3], a solution $u \in L^\infty((0, 1))$ of (P_λ) with $\Phi(w) \leq u \leq 0$. Now, to prove that (P_λ) admits a maximal solution, we consider u_1 the solution of

$$\begin{cases} S_k(D^2u_1) = \lambda|x|^\sigma & \text{in } B, \\ u_1 = 0 & \text{on } \partial B. \end{cases}$$

Since u is in particular a subsolution of (P_λ) , we have $u \leq u_1$ on B by the comparison principle, see [42]. Next, we define u_i ($i = 2, 3, \dots$) as the solution of

$$\begin{cases} S_k(D^2u_i) = \lambda|x|^\sigma (1 - u_{i-1})^q & \text{in } B, \\ u_i = 0 & \text{on } \partial B. \end{cases}$$

Using again the comparison principle we obtain a decreasing sequence of u_i bounded from below by u and by 0 from above. Hence, we can pass to the limit and we obtain a solution u_{max} of (P_λ) , which is maximal since the recursive sequence $\{u_i\}$ does not depend on the subsolution u . Now, let $\lambda_1 < \lambda_2$ and $u_{\lambda_1}, u_{\lambda_2}$ be maximal solutions of (P_{λ_i}) ($i = 1, 2$), respectively. Note that u_{λ_2} is a subsolution of (P_{λ_1}) , whence $u_{\lambda_2} \leq u_{\lambda_1}$ by the maximality of u_{λ_1} . \square

For $R > 1$, let B_R be a ball centered at zero with radius R such that $\overline{B} \subset B_R$ and let η be the solution of

$$\begin{cases} S_k(D^2\eta) = 1 & \text{in } B_R, \\ \eta = 0 & \text{on } \partial B_R. \end{cases}$$

Then there exists a negative constant β such that $\eta < \beta < 0$ on ∂B . Set $M = \max_{x \in \overline{B}} |\eta|$ and take $\lambda < (1 + M)^{-q}$. Then

$$S_k(D^2\eta) = 1 > \lambda(1 + M)^q \geq \lambda(1 - \eta)^q \geq \lambda|x|^\sigma(1 - \eta)^q \text{ in } B.$$

By [50, Theorem 3.3], for any $\lambda \in (0, (1 + M)^{-q})$ there exists a solution u_λ of (P_λ) . Thus we may define

$$\lambda^* = \sup\{\lambda > 0 : \text{there is a solution } u_\lambda \in C^2(B) \text{ of (5)}\}. \quad (35)$$

Hence $\lambda^* > 0$.

To see that λ^* is finite we consider the inequality

$$\Delta u \geq C(n, k)[S_k(D^2u)]^{\frac{1}{k}}, \quad u \in \Phi^k(B). \quad (36)$$

See e.g. [51]. Consider the eigenvalue problem

$$(E_m) \quad \begin{cases} -\Delta u = \lambda m(x)u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $m(x) := |x|^{\frac{\sigma}{k}}$. It is known that problem (E_m) has a first eigenvalue, $\lambda_{1,m} > 0$, associated with an eigenfunction $\phi_{1,m} > 0$. Let $\lambda \in (0, \lambda^*)$ and let u be a solution of problem (P_λ) . Then, using (36), we obtain

$$\Delta u \geq C(n, k)\lambda^{\frac{1}{k}}|x|^{\frac{\sigma}{k}}(1 - u)(1 - u)^{\frac{q-k}{k}} \geq C(n, k)\lambda^{\frac{1}{k}}|x|^{\frac{\sigma}{k}}(1 - u) \geq C(n, k)\lambda^{\frac{1}{k}}|x|^{\frac{\sigma}{k}}(-u),$$

which in turn implies $\lambda < \left(\frac{\lambda_{1,m}}{C(n, k)}\right)^k$. Thus λ^* is finite.

Now, let $\lambda \in (0, \lambda^*)$. Then u_λ is a maximal bounded solution of (P_λ) by Lemma 3.2 applied with $\lambda_0 \in (\lambda, \lambda^*)$.

Now let λ_i be an increasing sequence such that $\lambda_i \rightarrow \lambda^*$ as $i \rightarrow +\infty$ and let u_{λ_i} be a maximal solution of (P_{λ_i}) . By Lemma 3.2, for all $r \in [0, 1]$, we have $u_{\lambda_{i+1}}(r) \leq u_{\lambda_i}(r) \leq 0$. On the other hand, integrating the equation in (P_{λ_i}) , we obtain

$$u_{\lambda_i}(r) = \int_r^1 \left[c_{n,k}^{-1} \tau^{k-n} \int_0^\tau s^{n-1+\sigma} \lambda_i (1 - u_{\lambda_i}(s))^q ds \right]^{\frac{1}{k}} d\tau.$$

Now, applying twice the monotone convergence theorem we conclude that

$$u^*(r) := \lim_{i \rightarrow +\infty} u_{\lambda_i}(r), \quad \text{exists a.a. } r \in (0, 1)$$

and

$$u^*(r) = \int_r^1 \left[c_{n,k}^{-1} \tau^{k-n} \int_0^\tau s^{n-1+\sigma} \lambda^* (1 - u^*(s))^q ds \right]^{\frac{1}{k}} d\tau, \quad \text{a.a. } r \in (0, 1).$$

The assertion concerning the non existence of solutions follows directly from the definition of λ^* .

4 Classification of our Lotka-Volterra system

We use the notation of [38] to define

$$\begin{aligned} P(x, y) &= (n + \sigma)x - x^2 - qxy \\ Q(x, y) &= -\frac{n - 2k}{k}y + \frac{1}{k}xy + y^2 \end{aligned}$$

Now in order to compute the Poincaré index of a (finite) critical point (x_0, y_0) we define

$$\Lambda(x_0, y_0) = \partial_x P(x_0, y_0) \partial_y Q(x_0, y_0) - \partial_y P(x_0, y_0) \partial_x Q(x_0, y_0).$$

For system (14) we have that the critical points $(0, 0)$, $(n + \sigma, 0)$ and $(0, (n - 2k)/k)$ are saddle points with Poincaré index -1 . The fourth critical point (\hat{x}, \hat{y}) is an antisaddle with Poincaré index 1 . According to [38, Section 2.3.1] the notation for the saddle points is e^{-1} and for antisaddle points e^1 . Thus system (14) admits the combination $e^{-1}e^{-1}e^{-1}e^1$ of the finite critical points.

Next, in order to see the behavior of system (14) near infinity we use polar coordinates to compute the critical points at infinity. System (14) can be written as follows

$$r' = A_1(\theta) + rB_1(\theta) \quad (37)$$

$$\theta' = A_2(\theta) + rB_2(\theta). \quad (38)$$

Since the critical points at infinity are characterized by the condition $\theta' = 0$ (see [38, Chapter 1]). The dominant term at large r is $B_2(\theta)$ which is given by $B_2(\theta) = \cos \theta \sin \theta [(1/k + 1) \cos \theta + (q + 1) \sin \theta]$. The solutions of $B_2(\theta) = 0$ are given by $\theta = 0(\pi)$, $\theta = \pi/2(3\pi/2)$, and $\theta = \arctan(-(k + 1)(q + 1))$. Actually, these solutions correspond (modulo π) to three infinite critical points. Now, we use the Poincaré sphere. For this, we introduce the change of variable $z = 1/x$ and $u = y/x (= \tan \theta)$. Thus system (14) becomes

$$zz' = -z[-1 + (n + \sigma)z - qu] =: \hat{P}(z, u) \quad (39)$$

$$zu' = \frac{k + 1}{k}u - \left(n + \sigma + \frac{n - 2k}{k}\right)zu + (q + 1)u^2 =: \hat{Q}(z, u). \quad (40)$$

The critical points at infinity are given by $(0, u)$, where u is a solution of

$$\frac{k + 1}{k}u + (q + 1)u^2 = 0. \quad (41)$$

The eigenvalues of the linearized system (39)-(40) near infinity are given by

$$\begin{aligned} \lambda_z &= \partial_z \hat{P}(0, u) = 1 + qu \\ \lambda_u &= \partial_u \hat{Q}(0, u) = \frac{k + 1}{k} + 2qu, \end{aligned}$$

where $u = 0$, $u = -(k + 1)/(k(q + 1))$ are the solutions of (41). Defining Λ_c as the product of the eigenvalues of the linearized system (39)-(40), it is easy to see that $\Lambda_c > 0$ for all solution u of (41). For the third infinite critical point located on the y -axis at infinity, we rotate the axes such that the critical point under consideration is at the end of the x axis so that $\Lambda_c = -1(-1 - 1/k) > 0$, cf. [38, Page 27]. Hence system (14) admits the combination $E^1 E^1 E^1$ of infinite critical points, where E^1 denotes a node. See [38, Section 2.3.2] for a more general presentation. Hence our system is classified in the class $e^{-1}e^{-1}e^{-1}e^1 E^1 E^1 E^1$. This combination is studied in [38, Sections 3.4.1 and 11.2.2]. See [38, Section 11.2.2, figure 11.5 (a)-(b)] for the corresponding phase portraits.

Since the center case $q = q^*(k, \sigma)$ was already studied in Section 2, we focus only on orbits connecting the critical points $(n + \sigma, 0)$ and the point (\hat{x}, \hat{y}) located at the interior of the first quadrant.

5 Local analysis at the point (\hat{x}, \hat{y})

This section contains the hardest part of this work. The main difficulties here are to obtain the critical exponent $q_{JL}(k, \sigma)$ and to define an auxiliary variable a_σ defined below, which is the key to determinate the stability of our system.

Next we show that the orbits of system (14) start from $(n + \sigma, 0)$ and end at (\hat{x}, \hat{y}) . By (12) and (31), the function $y = y(t)$ satisfies

$$\lim_{t \rightarrow -\infty} y(t) = \lim_{r \rightarrow 0} r \frac{v'(r)}{-v(r)} = 0 \quad (42)$$

and for $x = x(t)$, we have $\lim_{t \rightarrow -\infty} x(t) = \lim_{r \rightarrow 0} \bar{\lambda}(k, \sigma) [-v(r)]^q \frac{r^{k+\sigma}}{[v'(r)]^k}$. Now, by (11) and L'hospital's rule, we have

$$\lim_{r \rightarrow 0} \frac{[v'(r)]^k}{r^{k+\sigma}} = \lim_{r \rightarrow 0} \frac{\bar{\lambda}(k, \sigma) \int_0^r \tau^{n-1+\sigma} [-v(\tau)]^q d\tau}{r^{n+\sigma}} = \lim_{r \rightarrow 0} \frac{\bar{\lambda}(k, \sigma) r^{n-1+\sigma} [-v(r)]^q}{(n + \sigma) r^{n-1+\sigma}} = \frac{\bar{\lambda}(k, \sigma)}{n + \sigma}.$$

Then

$$\lim_{t \rightarrow -\infty} x(t) = \frac{\bar{\lambda}(k, \sigma)(n + \sigma)}{\bar{\lambda}(k, \sigma)} = n + \sigma. \quad (43)$$

From (42) and (43) we conclude that

$$\lim_{t \rightarrow -\infty} (x(t), y(t)) = (n + \sigma, 0). \quad (44)$$

Remark 5.1. The variables x and y in (12) are nonnegative since the unique global solution v of (31) is negative and increasing by Lemma 3.1, which in turn imply that our study of the phase portrait must to be restricted only to positive orbits starting from $(n + \sigma, 0)$ and ending in (\hat{x}, \hat{y}) by the previous section.

Since, we are looking for orbits that start from $(n + \sigma, 0)$ and end in (\hat{x}, \hat{y}) , we only need a local analysis at the critical point (\hat{x}, \hat{y}) . To this end, we consider the linearization of (14) at the critical point (\hat{x}, \hat{y}) . The Jacobian matrix at the point (\hat{x}, \hat{y}) is given by

$$J = \begin{pmatrix} -\frac{q(n-2k)-(n+\sigma)k}{q-k} & -\frac{q[q(n-2k)-(n+\sigma)k]}{q-k} \\ \frac{2k+\sigma}{k(q-k)} & \frac{2k+\sigma}{q-k} \end{pmatrix}$$

The eigenvalues of J are $\lambda_{\pm} = \frac{1}{2} \text{tr} J \pm \frac{1}{2} \sqrt{(\text{tr} J)^2 - 4 \det J}$, where $\text{tr} J = \frac{2k+\sigma-[q(n-2k)-(n+\sigma)k]}{q-k}$ and $\det J = \frac{(2k+\sigma)[q(n-2k)-(n+\sigma)k]}{k(q-k)}$. The discriminant Δ is given by

$$\Delta(a_\sigma) = (q-k)^{-2} \left\{ [(2k+\sigma) - a_\sigma]^2 - 4 \frac{(2k+\sigma)(q-k)}{k} a_\sigma \right\}, \quad (45)$$

where

$$a_\sigma := q(n-2k) - (n+\sigma)k. \quad (46)$$

The location of the eigenvalues λ_{\pm} on the complex plane is determined as follows:

- (i) If $(2k + \sigma) - a_\sigma > 0$ and $\Delta \geq 0$, then the eigenvalues λ_{\pm} are real positive numbers.
- (ii) If $(2k + \sigma) - a_\sigma > 0$ and $\Delta < 0$, then the eigenvalues λ_{\pm} are complex numbers with positive real part.
- (iii) If $(2k + \sigma) - a_\sigma < 0$ and $\Delta < 0$, then the eigenvalues λ_{\pm} are complex numbers with negative real part.

(iv) If $(2k + \sigma) - a_\sigma < 0$ and $\Delta \geq 0$, then the eigenvalues λ_\pm are negative real numbers.

(v) If $(2k + \sigma) - a_\sigma = 0$, then the eigenvalues λ_\pm are purely imaginary.

The case $(2k + \sigma) - a_\sigma > 0$ corresponds to instability, $(2k + \sigma) - a_\sigma < 0$ to stability, and for $(2k + \sigma) - a_\sigma = 0$ to a center. We point out that $(2k + \sigma) - a_\sigma = 0$ is equivalent to defining $q = q^*(k, \sigma)$. We first note that $(2k + \sigma) - a_\sigma < 0$ is equivalent to $q > q^*(k, \sigma)$ and by Lemma 3.1 there exists a unique classical global solution of (31). In case $(2k + \sigma) - a_\sigma > 0$ (i.e. $q < q^*(k, \sigma)$) we cannot claim the existence of a global solution of (31) and thus the discussion about this case is excluded. In the stability case (iii)-(iv), that is $q > q^*(k, \sigma)$, depending on the sign of the discriminant Δ we obtain two types of orbits a spiral or a stable node. Using the same arguments given in [39] ($\sigma = 0$) we can establish that for $q^*(k, \sigma) < q < q_{JL}(k, \sigma)$ we have a spiral and for $q_{JL}(k, \sigma) \leq q$ a stable node. In fact, solving the equation

$$n - 2k = f_{k,\sigma}(q), \quad (47)$$

where

$$f_{k,\sigma}(q) = \frac{2q(2k + \sigma)}{k(q - k)} + \frac{2(2k + \sigma)}{k} \sqrt{\frac{q}{q - k}} + \frac{(2k + \sigma)(k - 1)}{q - k},$$

in q , we obtain the unique solution $q_{JL}(k, \sigma)$ of (47) and replacing it in (46) and (45), gives $\Delta = 0$. Observe that, for $k = 1$ and $\sigma = 0$, $f_{1,0}(q)$ coincides with the function f introduced in [32]. See [39] for more details in case $\sigma = 0$.

Lemma 5.1. *Let $n > 2k + 8 + 4\sigma/k$ and $q \geq q_{JL}(k, \sigma)$. Then the unique solution $(x(t), y(t))$ of (14) coincides with the graph of an increasing function $y = y(x)$*

Proof. The orbit starts at the critical point $(n + \sigma, 0)$ with a slope given by

$$\gamma_s := -\frac{(n - 2k)q^*(k, \sigma)}{qk(n + \sigma)}, \quad (48)$$

by L'Hospital's rule.

Now we describe the behavior of the orbit $(x(t), y(t))$ near (\hat{x}, \hat{y}) as time goes to infity. To this end, we compute

$$\gamma := \lim_{t \rightarrow +\infty} \frac{y'(t)}{x'(t)}$$

again by L'Hospital's rule. The existence of the limit above is equivalent to solving the equation

$$\gamma^2 + \frac{2k + \sigma + a_\sigma}{qa_\sigma} \gamma + \frac{2k + \sigma}{qka_\sigma} = 0, \quad (49)$$

whose roots are given by

$$\gamma_\pm = -\frac{2k + \sigma + a_\sigma}{2qa_\sigma} \pm \frac{1}{2qa_\sigma} \sqrt{(2k + \sigma + a_\sigma)^2 - 4\frac{(2k + \sigma)q}{k} a_\sigma}.$$

Note that $(q - k)^2 \Delta(a_\sigma) = (2k + \sigma + a_\sigma)^2 - 4\frac{(2k + \sigma)q}{k} a_\sigma$, where $\Delta(a_\sigma)$ is as in (45). Hence the roots of (49) are real numbers if, and only if, $q \geq q_{JL}(k, \sigma)$. Further, γ_\pm are negative roots.

Next, consider a function g defined by

$$g(x) = d(n + \sigma - x)^\alpha, \quad x \in [\hat{x}, n + \sigma] \quad (50)$$

where the constants d and α are chosen such that the graph of g connects the points $(n + \sigma, 0)$ and (\hat{x}, \hat{y}) . Using $\hat{y} = g(\hat{x})$, we obtain

$$d = q^{-\alpha} \left(\frac{2k + \sigma}{q - k} \right)^{1-\alpha}.$$

Now, setting $g'(\hat{x}) = (\gamma_+ + \gamma_-)/2$, we have $\alpha = \frac{2k+\sigma+a_\sigma}{2a_\sigma}$. Note that $\alpha \in (1/2, 1)$ since $q \geq q_{JL}(k, \sigma)$ and $2k + \sigma - a_\sigma = (n - 2k)(q^*(k, \sigma) - q) < 0$.

We claim that $(x(t), y(t))$ remains below the curve $y = g(x)$ when $x \in (\hat{x}, n + \sigma)$. Indeed, we first note that the curve $(x(t), y(t))$ lies above the line $y = -x/q + (n + \sigma)/q$ since the slope $-1/q$ is bigger than γ_s in (48) and remains from below $(x, g(x))$ near $(n + \sigma, 0)$ on the phase plane.

Suppose now by contradiction that $(x(t), y(t))$ intersects the curve $y = g(x)$ in a point (x_0, y_0) at $t = t_0$. In this case we have two possibilities: the orbit $(x(t), y(t))$ remains on the graph of g for all $t \geq t_0$ and then $(x(t), y(t))$ arrives at (\hat{x}, \hat{y}) with the same slope that $y = g(x)$ at \hat{x} , which is impossible since $g'(\hat{x}) = (\gamma_+ + \gamma_-)/2 \neq \gamma_\pm$ because the inequality $q > q_{JL}(k, \sigma)$. The other case is that the orbit crosses the graph of g at point x_0 . In this case we have

$$g'(x_0) > \frac{y'(t_0)}{x'(t_0)}. \quad (51)$$

We now show that (51) is impossible. To this end, define the functions

$$\begin{cases} F_1(x, y) = x(n + \sigma - x - qy), \\ F_2(x, y) = y(-\frac{n-2k}{k} + \frac{1}{k}x + y), \end{cases} \quad (52)$$

for all $(x, y) \in [\hat{x}, n + \sigma] \times [0, \hat{y}]$. Note that

$$F_2(x_0, g(x_0)) - g'(x_0)F_1(x_0, g(x_0)) = F_2(x_0, y_0) - g'(x_0)F_1(x_0, y_0) < 0. \quad (53)$$

Next, set

$$h(x) = F_2(x, g(x)) - g'(x)F_1(x, g(x)), \quad x \in (\hat{x}, n + \sigma). \quad (54)$$

Then, by (50) and (52) we have

$$h(x) = c_1(n + \sigma - x)^\alpha + c_2 x(n + \sigma - x)^\alpha + c_3(n + \sigma - x)^{2\alpha} + c_4 x(n + \sigma - x)^{2\alpha-1},$$

where $c_1 := -\frac{n-2k}{k}d$, $c_2 := \frac{k\alpha+1}{k}d$, $c_3 := d^2$ and $c_4 := -\alpha qd^2$.

Using (52), we obtain

$$h'(\hat{x}) = \frac{qa_\sigma}{q-k} \left(\beta^2 + \frac{2k+\sigma+a_\sigma}{qa_\sigma} \beta + \frac{2k+\sigma}{kqa_\sigma} \right), \quad (55)$$

where $\beta = g'(\hat{x})$. Further, since $\beta = (\gamma_+ + \gamma_-)/2$ we have $h'(\hat{x}) \leq 0$ by (49), with the strict inequality if $q > q_{JL}(k, \sigma)$ and the equality if $q = q_{JL}(k, \sigma)$ holds. Now note that $h(n + \sigma) = 0$ since $\alpha \in (1/2, 1)$ and $h(\hat{x}) = 0$ by (54) and the equality $\hat{y} = g(\hat{x})$. The derivative of h may be written as

$$h'(x) = d(n + \sigma - x)^{\alpha-1} \rho(x) \quad (56)$$

being the function ρ given by

$$\rho(x) = c_5(n + \sigma - x) - \left(2 + \alpha x + \frac{\sigma}{k} \right) \alpha + c_6(n + \sigma - x)^\alpha + c_7 x(n + \sigma - x)^{\alpha-1},$$

where $c_5 := \alpha + (\alpha + 1)/k$, $c_6 := -\alpha(q + 2)d$ and $c_7 := \alpha(2\alpha - 1)qd$.

Now, in order to determine the growth of h near $n + \sigma$, we rewrite h' as

$$h'(x) = d(n + \sigma - x)^{1-\alpha} \rho(x) / (n + \sigma - x)^{2(1-\alpha)}$$

and conclude that

$$\lim_{x \uparrow n+\sigma} h'(x) = \infty, \quad (57)$$

since $\alpha \in (1/2, 1)$. Note that $\rho(\hat{x}) \leq 0$ by (55)-(56). In particular, $\rho(\hat{x}) = 0$ in the case $q = q_{JL}(k, \sigma)$ by (54) and the equalities $g'(\hat{x}) = \gamma_+ = \gamma_- = \gamma$. On the other hand, it is easy to see that

$$\lim_{x \uparrow n+\sigma} \rho(x) = \infty. \quad (58)$$

Computing $\rho'(x)$ we conclude that the equation $\rho'(x) = 0$ admits only one solution which is not a minimum by (58). Further, the equation $\rho(x) = 0$ has at most two solutions on $[\hat{x}, n + \sigma)$ since, if there is no solution, then $\rho > 0$ on $[\hat{x}, n + \sigma)$ by (58) and hence by (56) we deduce that h is increasing on $[\hat{x}, n + \sigma)$ contradicting $h(\hat{x}) = h(n + \sigma) = 0$. Now, if we have two solutions on $(\hat{x}, n + \sigma)$ then $\rho(\hat{x}) > 0$, which is a contradiction since $\rho(\hat{x}) \leq 0$. Hence the equation $\rho(x) = 0$ admits at most two solutions on $[\hat{x}, n + \sigma)$. In this case we see that h admits at most two critical points on $[\hat{x}, n + \sigma)$. But this is impossible since $h(x_0) < 0$ by (53)-(54) and (55) and (57). The proof is now complete since (51) is impossible. \square

6 Proof of Theorem 2.1

The first statement concerning the existence and nonexistence of solutions of (P_λ) follows from Lemma 3.1 and Lemma 3.2.

We show the uniqueness and the multiplicity of solutions of (P_λ) as follows: let v be the unique solution of (31). Let $t_0 \in \mathbb{R}$ be fixed. Set $s_0 = e^{t_0}$ and choose a negative constant A depending on s_0 such that

$$v(s_0) = -(1 - A)^{-1} \quad (59)$$

holds.

Now we recall the rescaling

$$s = \left(\frac{\lambda}{\tilde{\lambda}(k, \sigma)} \right)^{\frac{1}{2k+\sigma}} (1 - A)^{\frac{q-k}{2k+\sigma}} r, \quad r > 0,$$

used to obtain (31). Set $r = 1$ and choose a positive constant λ depending on s_0 such that

$$s_0 = \left(\frac{\lambda}{\tilde{\lambda}(k, \sigma)} \right)^{\frac{1}{2k+\sigma}} (1 - A)^{\frac{q-k}{2k+\sigma}}. \quad (60)$$

Next, we write the value $v(s_0)$ in terms of the point $(x(t_0), y(t_0))$ with $s_0 = e^{t_0}$ (see (32)) as follows

$$v(s_0) = -[s_0^{2k+\sigma} \tilde{\lambda}(k, \sigma)]^{-\frac{1}{q-k}} (x(t_0)y(t_0)^k)^{\frac{1}{q-k}}. \quad (61)$$

Note that $\lambda = c_{n,k}x(t_0)y(t_0)^k$ by (59)-(61). Further, the map $s_0 \mapsto \lambda$ has range $(0, \tilde{\lambda})$ by the continuity of the orbit, $\lim_{t \rightarrow -\infty} y(t) = 0$ and $\lim_{t \rightarrow +\infty} c_{n,k}x(t)y(t)^k = \tilde{\lambda}$. Now we define

$$u_\lambda(r) = 1 + (1 - A)v(s).$$

Note that $u_\lambda(0) = A$ and u_λ solves problem (P_λ) . Now we discuss the multiplicity and the uniqueness of solutions. We consider the line $y = \lambda\tau_\sigma/\tilde{\lambda}$. Then for each intersection between the line and the orbit, we obtain one and/or several times t_0 and then one and/or several s_0 as well. Further, if we assume that there exist two different points $t_1, t_2 \in \mathbb{R}$ such that at line $y(s_1) = y(s_2)$ holds, then following the same argument as before we find two different solutions of problem (P_λ) for the same λ by (59).

Proof of (I). In this case the orbit describes a spiral starting from $(n + \sigma, 0)$ and ending at (\hat{x}, \hat{y}) . Thus the line $y = \lambda\tau_\sigma/\tilde{\lambda}$ may intersect one or several times to the orbit. On the other hand, the line $y = \lambda\tau_\sigma/\tilde{\lambda}$ intersects the curve $(x(t), y(t))$ an infinite numbers of times if $\lambda = \tilde{\lambda}$ and finite but large numbers of times if λ is sufficiently close to but not equal to $\tilde{\lambda}$. Hence, according to the explication above we conclude the proof.

Proof of (II). Here we apply Lemma 5.1. In the case $q \geq q_{JL}(k)$, we see that the line $y = \lambda\tau_\sigma/\tilde{\lambda}$ intersects the orbit $(x(t), y(t))$ only one time for each time t_0 , which in turn implies that the problem $(P_{\lambda(s_0)})$ admits a unique solution $u_{\lambda(s_0)}$. Now, if $\tilde{\lambda} < \lambda^*$ we see that the solution $u_{\lambda(s_0)}$ is a maximal solution by Lemma 3.2, which is decreasing as $\lambda(s_0)$ increases. Since $\lambda(s_0) \rightarrow \tilde{\lambda}$ as $s_0 \rightarrow +\infty$, we conclude that $u_{\lambda(s_0)}(0) \rightarrow A(\infty) > -\infty$ as $s_0 \rightarrow +\infty$ since $\tilde{\lambda} < \lambda^*$, which is impossible by (59). Therefore, $\tilde{\lambda} = \lambda^*$. This completes the proof of Theorem 2.1.

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